CERTAIN TYPE OF SPECIAL FUNCTION AND GENERAL POLYNOMIALS ASSOCIATE BY PATHWAY FRACTION INTEGRAL OPERATOR

Dr. Hemlata Saxena  
Professor of Mathematics,  
Applied Science  
Department Career Point University, Kota

Himanshu Sharma  
Student of B.Tech  
(Electrical Engineering)  
Career Point University, Kota

Roopal Pancholi  
Student of B.Tech  
(Electrical Engineering)  
Career Point University, Kota

ABSTRACT:

In the present paper, we consider product of some special functions associated with the pathway functional integral operator. This operator generalizes of the classical Riemann-Liouville fractional integral operator. The object of the present paper is to study a pathway fractional integral operator associated with the pathway model and pathway probability density for certain product of special functions with general argument the results derived here are quite general in nature and their several known and new special cases are also obtained here.

Keywords: Pathway fractional integral operator, Fox's H-function, Generalized Mittag-Leffler function, G-function and a general class of polynomials.

1. INTRODUCTION

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. During the last three decades, Fractional calculus has been applied to almost every field of science, engineering and mathematics. Many applications of fractional calculus can be found in turbulence and fluid dynamics, Stochastic dynamical system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, non-linear biological systems, astrophysics. In recent years, several authors have made significant contributions in the field of fractional calculus.

Several definitions of the operators of the classical and generalized fractional calculus are already well known and widely used in the applications to mathematical models of fractional order. The most popular one is the Riemann-Liouville fractional integral operator. The Pathway fractional integral operator introduced by Nair [9] is defined in the following manner

\[ (P_0^{(n,a)}f)(x) = x^n \left[ \int_0^x \left[ 1 - \frac{a(1-a)t}{x} \right]^\frac{\alpha}{1-\alpha} f(t) \, dt \right] \cdots (1.1) \]

where \( f(x) \in L(a, b), \eta \in C, R(\eta) > 0, a > 0 \) and 'pathway parameter' \( \alpha < 1 \).

The pathway model introduced by Mathai [6] and further studied by the Mathai and Haubold [7], [8]. For real scalar, the pathway model for scalar random variables is represented by the following probability density function (p. d. f.):

\[ f(x) = c |x|^{\gamma-1} [1 - \alpha(1 - \alpha)|x|^\delta]^{\frac{\beta}{1-\alpha}} \cdots (1.2) \]

\[ -\infty < x < \infty, \delta > 0, \beta > 0, [1 - \alpha(1 - \alpha)|x|^\delta]^{\frac{\beta}{1-\alpha}}, \gamma > 0 \]

where C is the normalizing constant and \( \alpha \) is called the pathway parameter. For real \( \alpha \), the normalizing constant is as follows:
\[
\begin{align*}
c &= \frac{1}{2} \delta \left[ a(1 - \alpha) \frac{\Gamma(\frac{\beta}{\gamma})}{\Gamma(\delta)} + \frac{\beta}{\gamma - 1} + 1 \right], \quad \alpha < 1 \\
&= \frac{1}{2} \delta \left[ a(1 - \alpha) \frac{\Gamma(\frac{\beta}{\gamma})}{\Gamma(\delta)} \right],
\end{align*}
\]

for \( \frac{1}{\alpha - 1} - \frac{\gamma}{\beta} > 0, \alpha > 1 \) \ldots (1.4)

\[
= \frac{1}{2} \frac{\Gamma(\frac{\beta}{\gamma})}{\Gamma(\delta)}, \quad \text{for} \quad \to 1 \quad \ldots (1.5)
\]

for \( \alpha < 1 \) it is a finite range density with \( 1 - \alpha(1 - \alpha)|x|^{-\beta}|x - \delta|^{-1} \) and (1.2) remains in the extended generalized type-1 beta family.

\[
f(x) = c|x|^{\gamma-1}[1 + a(\alpha - 1)|x|^{\delta}]^{\frac{\beta}{\alpha - 1}}
\]

\ldots (1.6)

Provided that \(-\infty < x < \infty, \delta > 0, \beta \geq 0, \alpha > 1 \) which is the extended generalized type-2 beta model for real \( x \). It includes the type-2 beta density, the F-density, the Cauchy density and many more.

Here we consider only the case of pathway parameter \( \alpha < 1 \). For \( \alpha \to 1 \), (1.2) and (1.6) take the exponential from, since

\[
\lim_{\alpha \to 1} c|x|^{\gamma-1}[1 - \alpha(1 - \alpha)|x|^{\delta}]^{\frac{\beta}{\alpha - 1}} = \lim_{x \to 1} c|x|^{\gamma-1}[1 + a(\alpha - 1)|x|^{\delta}]^{\frac{\alpha}{\alpha - 1}} = c|x|^{\gamma-1}e^{-\alpha|\gamma|} |x|^{\delta} \ldots (1.7)
\]

When \( \alpha \to 1 \), \[ 1 - \frac{a(1 - \alpha)x}{x} \to \frac{a}{x} \] \( U \), the operator (1.1) reduces to the Laplace integral transform of \( f \) with parameter \( \frac{an}{x} \):

\[
\left( \mathcal{L}_{0+}^{(n, \alpha)} f \right)(x) = x^\eta \int_0^\infty e^{-\eta x} f(t) dt \left| x \right|^{\alpha n} \ldots (1.8)
\]

when \( \alpha = 0, a = 1 \) then replacing \( \eta \) by \( \eta - 1 \) in (1.1) the integral operator reduces to the Riemann-Liouville fractional integral operator.

The following generalized M-series was introduced by Sharma and Jain [10]:

\[
\alpha', \beta'_{\rho M \sigma} \left( a_1, \ldots, a_\rho ; b_1, \ldots, b_\sigma ; z \right) = M(z)
\]

The pathway density in (1.2) for \( \alpha < 1 \), includes the extended type-1 beta density, the triangular density, the uniform density and many other p.d.f. for \( \alpha > 1 \), we have

\[
\alpha', \beta'_{\rho M \sigma} (z) = \sum_{k=0}^{\infty} (a_1', \ldots, a_\rho', k) \Gamma(\alpha k + \beta) Z^k = \psi_1(k) \ldots (1.9)
\]

where \( z, \alpha', \beta' \in c, Re(\alpha') > 0, \forall z i f \rho \leq \sigma, |z| < \alpha' \beta' \) for other details see [11].

Fox H-function [4] was studied by Skibiński [13] and defined as a following manner:

\[
H_{P, Q}^{M, N}[Z] = H_{P, Q}^{M, N} \left[ \frac{e_p}{e_p}, e_p \right] \left[ f_{Q}, F_{Q} \right] = \sum_{h=1}^{N} \sum_{v=0}^{\infty} (-1)^v X(\xi) (1) Z^\xi \ldots (1.10)
\]

where \( \xi = \frac{\epsilon_h + 1 - v}{v} \) and \( h=1, 2, \ldots \ldots \ldots \).

\[
\text{International Journal of Engineering Technology, Management and Applied Sciences}
\]

\[
\text{March 2017, Volume 5 Issue 3, ISSN 2349-4476}
\]

Dr.Hemlata Saxena, Himanshu Sharma, Roopal Pancholi
X(\xi) = \psi_2(\xi)

for convergence condition and other details see [4], [11].

A generalized Mittag-Leffler function studied by Shukla & Prajapati [12] in the following manner:

\[ E_{\alpha,\beta}^{p,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma qn + \beta)(z)^n}{\Gamma(\alpha n + \beta)} n! \]

\( \rho, \beta, \gamma \epsilon C, R(\rho) > 0, R(\beta) > 0 \) ... (1.11)

Where \( \alpha, \beta, \gamma \in C, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0 \) and \( q \in (0,1) \cup N \). This is a generalization of the exponential function \( \exp(z) \), the confluent hyper geometric function \( \phi(\gamma, \alpha, z) \), the Mittag-Leffler function \( E_{\alpha,\beta}(z) \), the Wirnens’s function \( E_{a,\beta}(z) \) and the function \( E_{a,\beta}(z) \) defined by Prabhakar.

\[ (\gamma)^q_n = \frac{\Gamma(\gamma + qn)}{\Gamma(\gamma)} \]

Denotes the generalized Pochhammer symbol which in particular reduces to \( q^n \prod_{r=1}^{q} \frac{n+r-1}{q} \) \( n \) if \( q \epsilon N \).

The G- and Hartley [5]:

\[ G_{\alpha,\mu,\gamma}[a, z] = z^{r+\mu-1} \sum_{n=0}^{\infty} \frac{(r)_n (\alpha z)^n}{\Gamma(1+n) \Gamma(nq + r\mu)} \]

... (1.12)

Srivastava [13] introduced the general class of polynomials

\[ S^m_n[x] = \sum_{l=0}^{n[m]} \frac{(-1)^{m-n}}{l!} A_{n,l} x^l, \] ... (1.13)

when \( m \) is an arbitrary positive integer and the coefficients \( A_{n,l} \) \((n,l \geq 0)\) are arbitrary constants, real or complex.

2. Main Results

**Theorem 1**

Let \( \eta, \gamma, \delta, \beta, T_1, T_2 > 0, Re(\eta) > 0, Re(\gamma) > 0, Re(\omega) > 0, Re(\delta) > 0, \) \( \frac{\omega + \delta}{\omega} > \max \),

\[ [0, -Re(\omega)], b, c, eR, \alpha < 1, Re \left[ \omega + \delta f{\frac{l}{F}} \right] > 0, \]

\[ 0, Re \left[ \omega + \frac{b}{\delta f} \right] |arg c| < \frac{1}{2} \pi, |arg b| < \frac{1}{2} \pi, \rho \leq \sigma \text{ and } |d| < \alpha^{\alpha}, \beta^* \geq 0, j = 1, \ldots, q, \]

and the coefficients \( A_{n,l} \) \((n,l \geq 0)\) are arbitrary constants, real or complex.

Then

\[ p^{(q\gamma)}_{\alpha,\beta}(\omega, \delta) \rho_M \frac{[d\gamma^{\gamma}]}{\rho \gamma} \frac{H^{M,N}}{P^Q} \left[ \frac{e_{\omega, \delta}}{(f_{\omega, \delta})^L} \right] E_{\alpha,\beta}^{p,q}(\delta \gamma) S_{q}^{m}[d\gamma^{\gamma}] \]

\[ = \psi_1(k) \psi_2(l) \]

\[ \frac{d^k}{dx^k} \left[ x^{\gamma + \omega - \lambda \beta - \lambda \beta^*} \Gamma \left( 1 + \frac{\eta}{1 - \alpha} \right) \right] \]

\[ \frac{\Gamma(\gamma)(\alpha - \gamma)}{\Gamma(\gamma + \lambda \beta)} \]

\[ \frac{c\delta^{\lambda}}{\alpha(1 - \alpha)^{\beta}} \left[ \frac{(e_p, e_p)}{(f_q, f_q)} \right] \]

... (2.1)

**Proof:** Making use (1.9), (1.10), (1.11), (1.12) and (1.1) in the theorem 1 then interchanging the order of integration and summation by means of beta function we at once arrive at the desired result (2.1).

**Theorem 2**
Let \( \eta, \omega \in \mathbb{C}, Re(\beta) > 0, Re(\delta) > 0, Re \left( 1 + \frac{n}{1 - \omega} \right) > 0, Re(\rho) > 0, \alpha < 1, be R, \)

\[
Re \left[ \omega + \delta \frac{f}{F} \right] > 0, Re \left[ \omega + \rho \frac{b}{B} \right] > 0 \arg c
\]

\[
< \frac{1}{2} T_1 \pi, \rho \leq \sigma, |arg b| < \frac{1}{2} T_2 \pi,
\]

\[
\beta^* > 0, T_1, T_2 > 0, \rho \leq \sigma, |d| < \frac{\alpha'}{\alpha}, j = 1, \ldots, Q, j' = 1, \ldots, q \text{ and the coefficients }
\]

\( A_{n, j} (n, j \geq 0) \) are arbitrary constants, real or complex.

Then

\[
ed^{(d)}(d')^{\alpha + \omega - \beta' k + q r - \delta - \beta' j} \Gamma \left( 1 + \frac{\eta}{1 - \alpha} \right)
\]

\[
\psi_2(l) \psi_1(k)
\]

\[
H^M_{P, Q} \left[ \frac{cx^\delta}{\alpha(1 - \alpha)^\delta} \right] \left( e_p, F_q \right)
\]

\[
= \psi_2(l) \psi_1(k)
\]

\[
H^M_{P, Q} \left[ \frac{cx^\delta}{\alpha(1 - \alpha)^\delta} \right] \left( e_p, F_q \right)
\]

\[
\frac{\psi_{n+1}^{(d)}(d')^{\alpha + \omega - \beta' k + q r - \delta - \beta' j} \Gamma \left( 1 + \frac{\eta}{1 - \alpha} \right)}{\left( \psi_{n+1}^{(d - 1)}(d')^{\alpha + \omega - \beta' k + q r - \delta - \beta' j} \Gamma \left( 1 + \frac{\eta}{1 - \alpha} \right) \right)}
\]

\[
\frac{\psi_{n+1}^{(d)}(d')^{\alpha + \omega - \beta' k + q r - \delta - \beta' j} \Gamma \left( 1 + \frac{\eta}{1 - \alpha} \right)}{\left( \psi_{n+1}^{(d - 1)}(d')^{\alpha + \omega - \beta' k + q r - \delta - \beta' j} \Gamma \left( 1 + \frac{\eta}{1 - \alpha} \right) \right)}
\]

\[
\frac{\psi_{n+1}^{(d)}(d')^{\alpha + \omega - \beta' k + q r - \delta - \beta' j} \Gamma \left( 1 + \frac{\eta}{1 - \alpha} \right)}{\left( \psi_{n+1}^{(d - 1)}(d')^{\alpha + \omega - \beta' k + q r - \delta - \beta' j} \Gamma \left( 1 + \frac{\eta}{1 - \alpha} \right) \right)}
\]

\[
\frac{\psi_{n+1}^{(d)}(d')^{\alpha + \omega - \beta' k + q r - \delta - \beta' j} \Gamma \left( 1 + \frac{\eta}{1 - \alpha} \right)}{\left( \psi_{n+1}^{(d - 1)}(d')^{\alpha + \omega - \beta' k + q r - \delta - \beta' j} \Gamma \left( 1 + \frac{\eta}{1 - \alpha} \right) \right)}
\]

Special Cases:

1. If we take polynomial \( S_n^m [\ast] \) is unity function

   \( \beta^* \to 0, \delta \to 0, \omega \to \rho \) e and \( k=1 \) in theorem-1 then we get the known result of [10, eq. (2.1), pp 14].

2. If we take polynomial \( S_n^m [\ast] \) is unity function

   \( k=1 \) \( \alpha \to \beta \) and \( \beta \to \omega \) in theorem-1 then we get the known result of [3, eq. (2.2)].

3. If we take polynomial \( S_n^m [\ast] \) is unity function

   \( k=1 \) \( \alpha \to \beta \) and \( \beta \to \omega \) in theorem-2 then we get the known result of [10, eq. (2.2), pp 14].

4. If we take polynomial \( S_n^m [\ast] \) is unity function

   \( \beta^* \to 0, \delta \to 0, \omega \to \rho \) and \( \gamma = k = 1 \) in theorem-1 then we get the known result of [9, eq. (25) p.244].

5. If we take polynomial \( S_n^m [\ast] \) is unity function

   \( \beta^* \to 0, \delta \to 0, \omega \to \rho \) and \( \gamma = k = 1 \) in theorem-1 then we get the known result of [9, eq. (26) p.245].

Conclusion: The pathway fractional integral operator is expected to have wide application in statistical distribution theory. It could help to extend some classical statistical distribution to wider classes of distribution. During the last three decades’ fraction calculus, has been applied to almost every field of science, engineering and mathematics.

Applications: Fractional calculus can be found in turbulence and fluid dynamics, Stochastic dynamical system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, non-linear biological systems, astrophysics.

References:

[2] Chaurasia, V.B.L. and Gill, Vinod, “Pathway fractional integral operator involving H-functions (Communicated).”


